A Weighted Sum Based Construction of PAC Codes

Wenxin Liu, *Member, IEEE*, Li Chen, *Senior Member, IEEE*, and Xingcheng Liu, *Senior Member, IEEE*

***Abstract*—The polarization-adjusted convolutional (PAC) codes concatenate an outer convolutional transform with an inner polar transform to improve the error-correction performance of polar codes. In the short-to-medium codeword length regime, it can approach the normal approximation (NA) bound. However, its optimal rate-profile remains unknown. This letter proposes a new construction for PAC codes based on a weighted sum (WS) metric, which considers both the cutoff rates and the utilizations of the polarized subchannels. In comparison with the existing Reed-Muller (RM) rate-profiles, it can yield a flexible rate. By adjusting the design signal-to-noise ratio (SNR), an appropriate tradeoff between the decoding performance and complexity can be achieved. Simulation results show the designed codes can outperform the ones that are designed by the existing techniques.**

***Index Terms*—Polarization-adjusted convolutional codes, polarized subchannels, rate-profile, weighted sum.**

I. Introduction

POLAR codes have been proven to be capacity achieving and their successive cancellation (SC) decoding exhibits a low complexity of $O(N\log_2 N)$, where N is the codeword length [1]. However, over the short-to-medium length regime, polar codes cannot achieve the finite-length bound, which also results in SC decoding being suboptimal. By keeping multiple decoding paths, the SC list (SCL) decoding was proposed to approach the maximum likelihood (ML) decoding performance with a complexity of $O(LN\log_2 N)$, where L is the decoding output list size [2]. Further assisted by the cyclic redundancy check (CRC) codes, the error-correction performance of polar codes can be significantly enhanced [2], [3]. The precoded short polar codes can also provide an enhanced SC or SCL decoding performance [4], [5].

Recently, the polarization-adjusted convolutional (PAC) codes [6] were proposed as a competent short-to-medium length code. It has been shown that a length-128 rate-$1/2$ PAC code can approach the normal approximation (NA) bound by using the Fano decoding [7]. Other decoding approaches such as the SCL decoding and the Viterbi list decoding have been investigated for PAC codes in [8], [9], and [10], respectively. With a sufficiently large list size, they can both approach the Fano decoding performance with a lower latency.

Manuscript received 24 June 2022; revised 5 August 2022; accepted 9 September 2022. Date of publication 22 September 2022; date of current version 9 January 2023. This work was supported in part by the National Natural Science Foundation of China under Grant 62071498, 61873290, and Grant 61972431; in part by the Key Project of the NSFC-Guangdong Province Joint Program, China, under Grant U2001204; and in part by the Science and Technology Program of Guangzhou, China, under Grant 202002030470. The associate editor coordinating the review of this letter and approving it for publication was M. Baldi. *(Corresponding author: Li Chen.)*

The authors are with the School of Electronics and Information Technology, Sun Yat-sen University, Guangzhou 510006, China (e-mail: liuwx6@mail2.sysu.edu.cn; chenli55@mail.sysu.edu.cn; isslxc@mail.sysu.edu.cn).

Digital Object Identifier 10.1109/LCOMM.2022.3209082

Despite its error-correction performance potential, the performance wise optimal rate-profile construction for PAC codes remains unknown. Two existing constructions are based on the polar rate-profiles and the Reed-Muller (RM) rate-profiles [6]. The polar rate-profiles select the reliable subchannels as the information positions. This can be realized through density evolution with Gaussian approximation (DE/GA) [11]. But the designed codes may have a poor weight distribution, resulting in a poor Fano decoding performance. Alternatively, the RM rate-profiles improve the decoding performance, but yield a higher complexity and limited rate choices. By considering both the row weights of the polar generator matrix and the reliability of the polarized subchannels, the RM-polar rate-profiles offer more rate choices [12]. However, the decoding performances of the constructed codes still fall away from the NA bound for a wide range of code parameters. Besides, the approach of [13] can reduce the number of minimum weight codewords of PAC codes, resulting in an enhanced decoding performance. The Monte Carlo based construction [14] was also proposed to improve the error-correction performance. But this method remains empirical.

This letter proposes a more effective construction for PAC codes, which yields a flexible rate and a good decoding performance. The proposed design is realized by a weighted sum (WS) metric which considers both the cutoff rate and the utilizations of the subchannels, i.e., the amount of information bits that are transmitted through one subchannel. The unfrozen bits will be assigned sequentially according to the WS metrics that are updated iteratively. It can be seen as a generalization of the RM rate-profiles, as when the rates coincide, the proposed construction and the RM rate-profiles will yield the same code. By adjusting the design signal-to-noise ratio (SNR), a good tradeoff between the decoding performance and complexity can be achieved. Our simulation results also show that the designed PAC codes can substantially outperform the ones that are designed based on the existing techniques.

II. PAC Codes and Fano Decoding

This section provides the prerequisites for this work, including the PAC codes and its Fano decoding.

A. Polar Codes

Polar codes are founded based on channel polarization, which consists of channel combining and splitting. Let W: $\mathcal{X} \rightarrow \mathcal{Y}$ denote a binary input discrete memoryless channel (BI-DMC), with an input alphabet $\mathcal{X} \in \{0,1\}$ and an arbitrary output alphabet $\mathcal{Y} \in \mathbb{R}$. For channel combining, N independent and identical channels are combined to create a vector channel W_N: $\mathcal{X}^N \rightarrow \mathcal{Y}^N$, where $N = 2^n$ and $n \in \mathbb{N}$. Channel splitting then splits W_N into a set of N polarized subchannels $\mathrm{W}_N^{(i)}$: $\mathcal{X} \rightarrow \mathcal{Y}^N \times \mathcal{X}^{i-1}$, where $1 \leq i \leq N$.

Fig. 1. Block diagram of a PAC code.

Let $\mathbb{F}_2$ denote the binary field. For a length-N polar code, its generator matrix is defined as

$$\mathbf{G}_\mathrm{P} = \mathbf{B}\mathbf{F}^{\otimes n}, \tag{1}$$

where $\mathbf{B} \in \mathbb{F}_2^{N\times N}$ is an $N \times N$ bit-reversal permutation matrix, $\mathbf{F} = ((1,0),(1,1))^T \in \mathbb{F}_2^{2\times 2}$ is a kernel matrix and $\otimes$ denotes the Kronecker product.

B. PAC Codes

A PAC code concatenates an outer convolutional code (of rate 1) and an inner polar code, which is shown as in Fig.1. Since both the input and output of the inner and outer codes are of equal length, the inner and outer encoding are also called polar transform and convolutional transform, respectively.

Given a length-N binary vector $\underline{v} = (v_1, v_2, \ldots, v_N) \in \mathbb{F}_2^N$, let $\underline{v}_{i_1}^{i_2}$ denote its subvector $(v_{i_1}, v_{i_1+1}, \ldots, v_{i_2})$, where $1 \le i_1 < i_2 \le N$. Hence, $\underline{v}$ can also be denoted as $\underline{v}_1^N$. Let $\underline{m}_1^K \in \mathbb{F}_2^K$ and $\underline{s}_1^N \in \mathbb{F}_2^N$ denote the information vector and the input vector of the convolutional transform, respectively. To encode an (N,K) PAC code, $\underline{s}_1^N$ will be partitioned into the information set $\mathcal{A}$ and the frozen set $\mathcal{A}^c$, respectively, where $|\mathcal{A}| = K$ and $|\mathcal{A}^c| = N - K$. Subsequently, information bits of $\underline{m}_1^K$ will be placed in $\underline{s}_1^N$ at the positions indexed by $\mathcal{A}$. The rest positions indexed by $\mathcal{A}^c$ will be filled with frozen bits. In this work, the frozen bits are represented by zeros.

Let $g(x) = g_0 + g_1 x + \cdots + g_l x^l$ denote the generator polynomial of the convolutional encoder with a constraint length of $l+1$. The convolutional generator matrix can be written as an upper-triangular Toeplitz matrix $\mathbf{G}_\mathrm{c}$. The convolutional transform output $\underline{u}_1^N = (u_1, u_2, \ldots, u_N) \in \mathbb{F}_2^N$ is generated by

$$\underline{u}_1^N = \underline{s}_1^N \mathbf{G}_\mathrm{c}. \tag{2}$$

Further based on the generator matrix of the inner polar code, the codeword is generated by

$$\underline{c}_1^N = \underline{u}_1^N \mathbf{G}_\mathrm{p}, \tag{3}$$

where $\underline{c}_1^N = (c_1, c_2, \ldots, c_N) \in \mathbb{F}_2^N$. The PAC code can be regarded as transmitting a convolutional codeword through the polarized channel. Assuming a binary modulation, e.g., BPSK, is used for transmitting the codeword, let $\underline{y}_1^N \in \mathbb{R}^N$ denote the received vector. Further let $P(y|x)$ denote the channel transition probability, where $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. For each subchannel, its Bhattacharyya parameter is defined as

$$\begin{aligned} &Z(\mathrm{W}_N^{(i)}) \\ &= \sum_{\underline{y}_1^N \in \mathcal{Y}^N} \sum_{\underline{u}_1^{i-1} \in \mathcal{X}^{i-1}} \sqrt{P(\underline{y}_1^N, \underline{u}_1^{i-1}|0) P(\underline{y}_1^N, \underline{u}_1^{i-1}|1)}. \end{aligned} \tag{4}$$

For the additive white Gaussian noise (AWGN) channel, the Bhattacharyya parameter can be estimated as

$$Z\left(\mathrm{W}_N^{(i)}\right) = e^{\frac{-1}{2\left(\sigma_N^{(i)}\right)^2}}, \tag{5}$$

where $\left(\sigma_N^{(i)}\right)^2$ is the noise variance of the subchannel. It can be estimated via the Gaussian approximation (GA) [11]. The cutoff rate $E_0(1, \mathrm{W}_N^{(i)})$ can be further obtained by [15]

$$E_0(1, \mathrm{W}_N^{(i)}) = 1 - \log_2(1 + Z(\mathrm{W}_N^{(i)})), \tag{6}$$

which is also the lower bound for the mutual information of subchannel $\mathrm{W}_N^{(i)}$. It indicates if the convolutional code rate is above this value, the average Fano decoding complexity will become practically prohibitive.

C. Fano Decoding

Fano decoding was adopted to decode PAC codes in [6]. Representing the SC decoding over a binary tree, Fano decoding is a depth-first-search mechanism that prioritizes the finding of a complete root-to-leaf path, forming an estimated information sequence over the tree.

Let $\underline{\hat{u}}_1^N \in \mathbb{F}_2^N$ denote the estimated information vector of the inner polar code. The Fano decoding path metric can be computed as [16]

$$\begin{aligned} \mathcal{M}\left(\underline{\hat{u}}_1^i\right) &= \mathcal{M}\left(\underline{\hat{u}}_1^{i-1}\right) + 1 + \log_2 P(u_i|\underline{y}_1^N, \underline{\hat{u}}_1^{i-1}) \\ &\quad - E_0(1, \mathrm{W}_N^{(i)}), \end{aligned} \tag{7}$$

where $P(u_i|\underline{y}_1^N, \underline{\hat{u}}_1^{i-1})$ is the *a posteriori* probability of u_i. The decoding path moves from a node to either its ancestral nodes or its descendants. It visits a node only if its Fano path metric $\mathcal{M}\left(\underline{\hat{u}}_1^i\right)$ is not smaller than a threshold T. This threshold is dynamically adjusted with a step size of Δ, in order to ensure the decoding can yield a complete path.

III. The Weighted Sum Based Construction

This section proposes the new construction for PAC codes. The design considers both the subchannels weight and their utilization, which will be introduced as below.

A. Subchannel Weight

In the short-to-medium length regime, the capacity of some subchannels remain unpolarized. They are often assigned with the frozen bits, limiting transmission efficiency. PAC codes advance from polar codes by further utilizing those subchannels with the assistance of the convolutional transform.

To characterize the capacity of each subchannel and control the decoding complexity, the quantized cutoff rate of each subchannel is considered. They satisfy

$$\omega_i = \left\lceil \frac{E_0(1, \mathrm{W}_N^{(i)})}{0.1} \right\rceil, \tag{8}$$

where $0 \le \omega_i \le 10$. The subchannel weight vector $\underline{\omega}_1^N = (\omega_1, \omega_2, \ldots, \omega_N)$ can be further formed. The subchannel with a larger weight has a higher potential for transmitting information. The quantization of (8) reduces the discrepancy between the subchannel cutoff rates, so that their weights can be appropriately adjusted in the proposed WS metric. As a result, subchannels with a small cutoff rate but not used for transmitting information can be better utilized. Note that using the original subchannel cutoff rates of (6) results in the polar-like rate-profile.

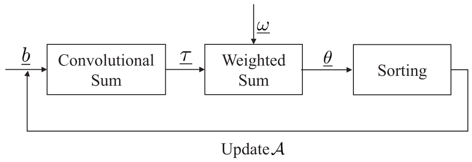

Fig. 2. Block diagram of the proposed construction.

B. Subchannel Utilization

Through the convolutional transform, part of the input vector $\underline{s}_1^N$ are convolutionally encoded. The coded bit u_i is generated by

$$u_i = s_i g_0 \oplus s_{i-1} g_1 \oplus \cdots \oplus s_{i-j} g_j \oplus \cdots \oplus s_{i-l} g_l. \quad (9)$$

Since the inputs of the convolutional transform are partitioned into the information set $\mathcal{A}$ and the frozen set $\mathcal{A}^c$, we let b_i denote the status of s_i. If $i \in \mathcal{A}$, $b_i = 1$; otherwise, $b_i = 0$. Let $\underline{b}_1^N = (b_1, b_2, \ldots, b_N)$. For each subchannel $\mathrm{W}_N^{(i)}$, the channel utilization τ_i is defined as

$$\tau_i \triangleq \sum_{j=0}^{l} b_{i-j} g_j. \quad (10)$$

It indicates the amount of information bits that are transmitted through subchannel $\mathrm{W}_N^{(i)}$. Subsequently, the channel utilization vector can be formed as $\underline{\tau}_1^N = (\tau_1, \tau_2, \ldots, \tau_N)$, where τ_i. A good code design should match the subchannel utilization to its condition.

C. The Proposed Construction

With the above mentioned subchannels weight and utilization, reliability of each position in $\underline{s}_1^N$ can be analyzed. For subchannel $\mathrm{W}_N^{(i)}$, a newly added information bit will share the subchannel with the other τ_i information bits. Therefore, its weight should be attenuated accordingly by $\omega_i/(\tau_i + 1)$. The convolutional transform enables an information bit to be transmitted through multiple polarized subchannels. Assuming s_i is an information bit, its WS metric is defined as

$$\theta_i \triangleq \sum_{j=0}^{l} \frac{g_j \omega_{i+j}}{\tau_{i+j} + 1}, \quad (11)$$

where $\omega_{i+j}/(\tau_{i+j} + 1) \triangleq 0$, $\forall i > N - j$. The WS vector $\underline{\theta}_1^N = (\theta_1, \theta_2, \ldots, \theta_N)$ can be further formed. A row that corresponds to a greater WS metric is considered as having a greater capability for transmitting information.

Let us further define the RM score function $q(i)$ as $q(i) = f(i-1)$ and $f(i-1)$ is the number of 1s in the binary representation of $i-1$, where $1 \leq i \leq N$. E.g., the binary representation of 5 is 101, and $f(5) = 2$. Given row i of $\mathbf{G}_\mathrm{P}$, its RM score can be determined as $q(i)$. Note that $2^{q(i)}$ is the weight of the row. The proposed construction is an iterative process that updates $\mathcal{A}$, which is shown as in Fig. 2. Given parameters N and K, a non-negative integer t that satisfies

$$\sum_{p=t+1}^{n} \binom{n}{p} \leq K < \sum_{p=t}^{n} \binom{n}{p} \quad (12)$$

is selected, where $n = \log_2 N$. The RM scores are utilized to determine the initial information set $\mathcal{A}$ and the optional information set $\mathcal{S}$. Set $\mathcal{A}$ collects the rows which satisfy $q(i) > t$ and $|\mathcal{A}| \leq K$, while set $\mathcal{S}$ collects the rows which satisfy $q(i) = t$. For long PAC codes, the decoding performance can be improved by easing the restriction of $\mathcal{A}$. If $|\mathcal{A}| = K$, the proposed rate-profile will be the same as the RM rate-profile. With the initial information set $\mathcal{A}$ and $|\mathcal{A}| < K$, the channel utilization vector $\underline{\tau}_1^N$ can be obtained as in (10). Given a design SNR, the subchannel weight vector $\underline{\omega}_1^N$ can be further obtained. Based on $\underline{\omega}_1^N$ and $\underline{\tau}_1^N$, the WS vector $\underline{\theta}_1^N$ can be obtained. Among its entries θ_i where $i \in \mathcal{S}$, the maximum one will be identified as

$$i^* = \arg\max\{\theta_i, i \in \mathcal{S}\}. \quad (13)$$

It will then be moved from $\mathcal{S}$ to $\mathcal{A}$. If there are more than one maximum values, the one with the smallest i will be selected. With an updated $\mathcal{A}$, vectors $\underline{\tau}_1^N$ and $\underline{\theta}_1^N$ will be further updated as in (10) and (11), respectively. Set $\mathcal{A}$ will be further renewed. This iterative process continues until $|\mathcal{A}| = K$. The following Algorithm 1 summarizes this iterative construction.

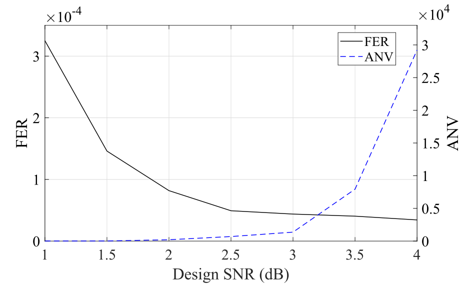

Fig. 3. Performance-complexity tradeoff of the (256, 128) PAC codes.

Algorithm 1 The Weighted Sum Based Construction

Input: N, K, design SNR;
Output: $\mathcal{A}$;
  Determine $\mathcal{A}$ and $\mathcal{S}$ based on the RM scores;
  If $|\mathcal{A}| < K$ **do**
    Obtain $E_0(1, \mathrm{W}_N^{(i)})$ by the GA;
    Determine $\underline{\omega}_1^N$ as in (8);
    While $|\mathcal{A}| < K$ **do**
      Update $\underline{b}_1^N$ and $\underline{\tau}_1^N$ as in (10);
      Update $\underline{\theta}_1^N$ as in (11);
      Identify i^* as in (13) and update $\mathcal{S}$ and $\mathcal{A}$;
    End While
  End If

D. Performance-Complexity Tradeoff

Similar to polar codes, the design SNR is critical for the code's performance. This also affects the decoding complexity. Hence, a PAC code is often designed at an SNR that can yield a good performance-complexity tradeoff. We define the average number of visits (ANV) as the average times that each coded bit (over the decoding tree) is visited by the Fano decoder [17]. It is a measure of the decoding complexity. Fig. 3 shows the decoding frame error rate (FER) and ANV performance tradeoff for the (256, 128) PAC codes that are designed over a range of SNR. The results were obtained over the AWGN channel with SNR = 2.5dB.

At a low design SNR, the information positions mostly reside in the polarized subchannels, forming a polar-like

TABLE I

RATE PROFILES OF THE WS BASED CONSTRUCTION

(N, K)	Design SNR (dB)	Rate Profile
(256, 128)	2	000000010001151700031 31F037F777F0017157F177F577F1777777F577F7FFF
	3	0000000100031717001 7115F1577577F0117157F1577577F1577577F577f7FFF
(128, 42)	2.5	00000001000715170017111F1577177F
(128, 85)	2.5	0003131F037F7F7F077F7F7F777F7FFF

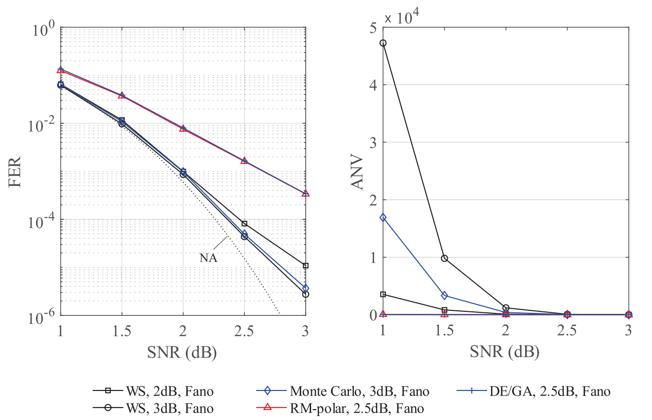

Fig. 4. Performance of the (256, 128) PAC codes.

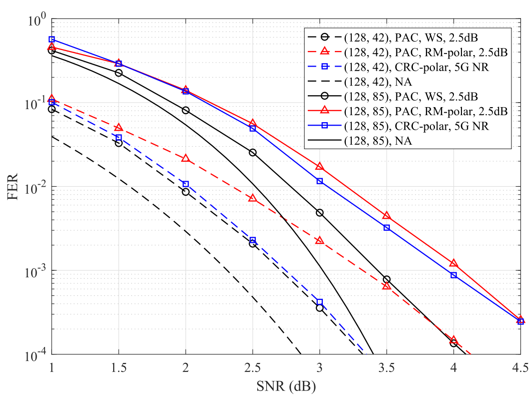

Fig. 5. Performance of different codes with $N = 128$ under SCL, $L = 32$.

construction. It has a low complexity but yields a poor decoding performance. As the design SNR increases, weights of the polarized subchannels that have a medium cut off rate will increase. They will then be considered for transmitting information bits. It improves the decoding performance, but at the cost of the decoding complexity. Fig. 3 shows a design SNR of 2dB and beyond will result in a decoding FER below 10^{-4}. Meanwhile, the corresponding ANV will increase rapidly after 3dB. Therefore, for the (256, 128) PAC codes, the design SNR region is 2-3 dB. PAC codes of other parameters can be designed similarly.

IV. SIMULATION RESULTS

PAC codes with the generator polynomial $g(x) = 1 + x^3 + x^7 + x^9 + x^{10}$ were simulated over the AWGN channel using BPSK. The Fano decoding functions with $\Delta = 2$, which was chosen by considering the performance-complexity tradeoff. The proposal is compared with the RM-polar construction [12], the DE/GA construction [11] and the Monte Carlo based construction [14]. The RM-polar and the DE/GA constructions were designed using GA at the SNR of 2.5dB, as in [14].

Table I shows the rate-profiles of the constructed codes. They are presented in the hexadecimal form, and the zeros indicate the frozen bits. Note that the RM construction cannot produce codes of these rates. Fig. 4 shows the FER and ANV performances of the (256, 128) PAC codes. It can be seen that the proposed construction yields an improved performance over the RM-polar and DE/GA construction. As the design SNR increases, the FER performance of the PAC codes improves but also incurring a greater decoding complexity. At the design SNR of 3dB, it slightly outperforms the Monte Carlo based construction but with a larger decoding ANV. However, the proposed design complexity is $O(N)$, which is much smaller than the Monte Carlo approach. Fig. 5 further compares the proposed PAC codes with the CRC-polar codes. The CRC-polar codes are the 5th generation new radio (5G NR) standard codes with a CRC generator polynomial of $g_c(x) = 1 + x + x^2 + x^8$. Both the WS and RM-polar design SNR are 2.5 dB. It can be seen that with the SCL decoding and $L = 32$, the proposed PAC codes can outperform the RM-polar constructed PAC codes and the 5G NR CRC-polar.

REFERENCES

[1] E. Arikan, "Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels," *IEEE Trans. Inf. Theory*, vol. 55, no. 7, pp. 3051–3073, Jan. 2009.
[2] I. Tal and A. Vardy, "List decoding of polar codes," *IEEE Trans. Inf. Theory*, vol. 61, no. 5, pp. 2213–2226, May 2015.
[3] K. Niu and K. Chen, "CRC-aided decoding of polar codes," *IEEE Commun. Lett.*, vol. 16, no. 10, pp. 1668–1671, Oct. 2012.
[4] P. Trifonov and V. Miloslavskaya, "Polar subcodes," *IEEE J. Sel. Areas Commun.*, vol. 34, no. 2, pp. 254–266, Feb. 2016.
[5] V. Miloslavskaya and B. Vucetic, "Design of short polar codes for SCL decoding," *IEEE Trans. Commun.*, vol. 68, no. 11, pp. 6657–6668, Nov. 2020.
[6] E. Arıkan, "From sequential decoding to channel polarization and back again," 2019, *arXiv:1908.09594*.
[7] R. Fano, "A heuristic discussion of probabilistic decoding," *IEEE Trans. Inf. Theory*, vol. IT-9, no. 4, pp. 64–74, Apr. 1963.
[8] H. Yao, A. Fazeli, and A. Vardy, "List decoding of Arıkan's PAC codes," *Entropy*, vol. 23, no. 7, p. 841, 2021.
[9] M. Rowshan, A. Burg, and E. Viterbo, "Polarization-adjusted convolutional (PAC) codes: Sequential decoding vs list decoding," *IEEE Trans. Veh. Technol.*, vol. 70, no. 2, pp. 1434–1447, Feb. 2021.
[10] M. Rowshan and E. Viterbo, "List Viterbi decoding of PAC codes," *IEEE Trans. Veh. Technol.*, vol. 70, no. 3, pp. 2428–2435, Mar. 2021.
[11] P. Trifonov, "Efficient design and decoding of polar codes," *IEEE Trans. Commun.*, vol. 60, no. 11, pp. 3221–3227, Nov. 2012.
[12] B. Li, H. Shen, and D. Tse, "A RM-polar codes," 2014, *arXiv:1407.5483*.
[13] M. Rowshan, S. H. Dau, and E. Viterbo, "Error coefficient-reduced polar/PAC codes," 2011, *arXiv:2111.088435*.
[14] M. Moradi and A. Mozammel, "A Monte–Carlo based construction of polarization-adjusted convolutional (PAC) codes," 2021, *arXiv:2106.08118*.
[15] R. G. Gallager, *Information Theory and Reliable Communication*. New York, NY, USA: Wiley, 1968.
[16] M. Moradi, "On sequential decoding metric function of polarization-adjusted convolutional (PAC) codes," *IEEE Trans. Commun.*, vol. 69, no. 12, pp. 7913–7922, Dec. 2021.
[17] J. M. Wozencraft, "Sequential decoding for reliable communication," Res. Lab. Electron., MIT, Cambridge, MA, USA, Tech. Rep. 325, Aug. 1957.